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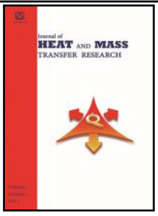

Non-Fourier heat conduction equation in a sphere; comparison of variational method and inverse laplace transformation with exact solution

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ABSTRACT

Small scale thermal devices, such as micro heater, have led researchers to consider more accurate models of heat in thermal systems. Moreover, biological applications of heat transfer such as simulation of temperature field in laser surgery are other pathways which urge us to re-examine thermal systems with modern ones. Non-Fourier heat transfer overcomes some shortcomings of Fourier heat transfer, when small scale systems are considered or non-homogeneous materials are under study. In this study, the hyperbolic heat conduction problem in a sphere is solved by three approaches. 1. Finding the exact solution using the method of separation of variables
2. Finding two approximate solutions using the Laplace transformation and then a. applying the variational method for finding the Laplace inverse b. finding the solution of the problem in Laplace domain and using an asymptotic series to evaluate the solution for small values of times

Various orders for the variational method are considered and compared against the analytical solution. Since the two latter methods can be used in nonlinear problems such as those including radiation heat losses, the approximate solutions can be useful additionally in the field of thermal analysis of non-Fourier problems.

1. Introduction

Fourier's law is usually used to describe steady and transient heat transfer problems to predict the temperature field in the objects. The methodology is accurate for engineering problems under regular conditions [1-2]. The main assumption in this theory is that the heat flux has a linear relation with the temperature gradient and the propagation speed of the thermal wave is infinite [1]. Consequently, any thermal disturbance exerted on a body is instantaneously felt through the whole of the body. However, during the past few decades, there have been some researches concerned with departures from the classical Fourier heat conduction law. The motivation for these researches was to eliminate the paradox of an infinite thermal wave speed. However the current trend in nanoscale devices and manufacturing processes has sparked and renewed the interest in hyperbolic conduction law, in order to eliminate this paradox [3]. Moreover, in situations which include extremely high temperature gradients, extremely large heat fluxes and extremely short transient duration or in the case of the near absolute zero temperature, the heat propagation speeds are finite and the mode of heat conduction is

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propagative and non-diffusive [4]. There have been numerous attempts to present a new heat conduction formulation. But it seems that the most frequently used model is the hyperbolic heat conduction introduced by Cattaneo [5] and Vernotte [6]. These two scholars considered a time-lag between the heat flux and temperature variation within the medium. This time-lag can be very short which occurs in metals [7] more than seconds which has been observed in biological tissues [8]. The non-Fourier effect becomes more and more attractive in practical engineering problems such as the non-homogenous-solid-conduction process [9], the rapid heating process [10], the slow-conduction process [7], etc. Recently, some new fundamental analyses have been done in a different configuration based on the non-Fourier heat conduction [11-13].

Due to the hyperbolic nature of this model, the solution procedure is challenging and more difficult compared to the Fourier model [14]. It has been stated in the literature that due to its mathematically ill-posed character, even the numerical solution is hard to obtain [15]. Therefore, every single novel attempt in this area is appreciated. Recently much more attention has been paid to some approximate analytical methods, including variational iteration method [16], homotopy perturbation method [17] and variational formulation method [18] to the solution of parabolic and hyperbolic heat transfer equations. When compared to other approximate analytical methods, variational methods combine the following two advantages [19]: (a) they provide physical insight into the nature of the solution of the problem; (b) the obtained solutions are the best among all the possible trial-functions. Therefore, the variational methods have been, and continue to be, popular tools for linear and nonlinear analysis. Arpaci et al. [20] eliminated the use of the penetration depth by considering the variational formulation of the Laplace-transform of unsteady diffusion problem. One of the advantages of this method is the profiles that are convenient for solving the transformed problem and generally yield a simple transformed solution that does not require the use of the inversion integral.

This study applies three approaches namely separation of variables as an exact solution, variational method and asymptotic series for taking the inverse Laplace as two approximate solutions. The employment of the Laplace transformation in the hyperbolic heat conduction equation leads to a second-order differential equation in the spatial variable. The transformed temperature profiles for the first fifth- order accuracy are obtained to illustrate the ability of the variational method. Also, using the asymptotic series in the solution of the problem in Laplace domain gives a high precision solution for small values of times. The close agreements between the exact values and the estimated results confirm the validity and the accuracy of the two approximate proposed methods.

2. Problem Statement

Consider the transition conduction in a sphere of radius *R*. The sphere is initially at a uniform temperature T_0. The outer surface of the sphere is suddenly raised to temperature T_∞. Assuming constant thermophysical properties and no internal heat generation the formulation of this problem is as follows [3]:

$$\alpha\left(\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}\right)=\tau_r\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t} \tag{1}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0}=0 \quad for\, t>0 \tag{2a}$$

$$T\big|_{r=1}=T_\infty \quad for\, t>0 \tag{2b}$$

$$T\big|_{t=0}=T_0 \quad for \quad 0<r<R \tag{2c}$$

$$\left.\frac{\partial T}{\partial t}\right|_{t=0}=0 \quad for \quad 0<r<R \tag{2d}$$

Where α is the thermal diffusivity and τ_r is the relaxation time. This relaxation time means that there is a time difference between the temperature gradient within the material and the applied heat flux. The more this relaxation time value is, the more would be the time difference between these two phenomena. For convenience in the subsequent analysis, we introduce the following dimensionless quantities:

$$\varepsilon=\frac{r}{R} \qquad \theta=\frac{T-T_0}{T_\infty-T_0}$$
$$\tau=\frac{\alpha t}{R^2} \qquad \tau'=\frac{\alpha\tau_r}{R^2} \tag{3}$$

Where ε, θ, τ and τ' are the non-dimensional radius, temperature, time and relaxation time, respectively. Introducing the dimensionless quantities, the normalized equation and boundary-initial conditions will be expressed as follows [3]:

$$\frac{\partial^2\theta}{\partial\varepsilon^2}+\frac{2}{\varepsilon}\frac{\partial\theta}{\partial\varepsilon}=\tau'\frac{\partial^2\theta}{\partial\tau^2}+\frac{\partial\theta}{\partial\tau} \tag{4}$$

$$\left.\frac{\partial\theta}{\partial\varepsilon}\right|_{\varepsilon=0}=0 \quad for \quad \tau>0 \tag{5a}$$

$$\theta\big|_{\varepsilon=1}=1 \quad for \quad \tau>0 \tag{5b}$$

$$\theta\big|_{\tau=0}=0 \quad for \quad 0<\varepsilon<1 \tag{5c}$$

$$\left.\frac{\partial\theta}{\partial\tau}\right|_{\tau=0}=0 \quad for \quad 0<\varepsilon<1 \tag{5d}$$

2.1. Exact solution (using separation of variables)

If we want to apply the well-known separation of variables method, first we should split up Equation (4) with the boundary and the initial conditions (5) into a set of simpler problems. Hahn and Özisik [21] suggested the following:

$$\theta(\varepsilon,\tau)=\varphi(\varepsilon)+\psi(\varepsilon,\tau) \tag{6}$$

Where φ is taken as the solution of the following problem

$$\frac{\partial^2\varphi}{\partial\varepsilon^2}+\frac{2}{\varepsilon}\frac{\partial\varphi}{\partial\varepsilon}=0 \tag{7}$$

$$\left.\frac{\partial\varphi}{\partial\varepsilon}\right|_{\varepsilon=0}=0 \tag{8a}$$

$$\left.\varphi\right|_{\varepsilon=1}=1 \tag{8b}$$

And ψ is taken as the solution of the following problem

$$\frac{\partial^2\psi}{\partial\varepsilon^2}+\frac{2}{\varepsilon}\frac{\partial\psi}{\partial\varepsilon}=\tau'\frac{\partial^2\psi}{\partial\tau^2}+\frac{\partial\psi}{\partial\tau} \tag{9}$$

$$\left.\frac{\partial\psi}{\partial\varepsilon}\right|_{\varepsilon=0}=0 \tag{10a}$$

$$\left.\psi\right|_{\varepsilon=1}=0 \tag{10b}$$

$$\left.\psi\right|_{\tau=0}=-\varphi \tag{10c}$$

$$\left.\frac{\partial\psi}{\partial\tau}\right|_{\tau=0}=0 \tag{10d}$$

Solving Eq. (7) we obtain the solution of the steady problem as follows:

$$\varphi=\frac{C}{\varepsilon}+K \tag{11}$$

Applying Eqs. (8) yields $C=0$ and $K=1$. Then, we have the following:

$$\varphi=1 \tag{12}$$

Using the separation of variables method and applying $\psi=A(\varepsilon)B(\tau)$ to Eq. (9) we obtain the following:

$$\frac{A''}{A}+\frac{2}{\varepsilon}\frac{A'}{A}=\tau'\frac{B''}{B}+\frac{B'}{B}=\pm\lambda^2 \tag{13}$$

Here $-\lambda^2$ is suitable for our problem. Finally, the problem can be expressed separately in the ε and τ coordinates as follows:

$$A''+\frac{2}{\varepsilon}A'+\lambda^2A=0 \tag{14a}$$

$$\tau'B''+B'+\lambda^2B=0 \tag{14b}$$

And the homogeneous boundary and initial conditions are expressed as follows:

$$A'(0)=0 \tag{15a}$$

$$A(1)=0 \tag{15b}$$

$$B'(0)=0 \tag{15c}$$

Solving the Eq. (14a) yields the following:

$$A(\varepsilon)=C_1\frac{\sin(\lambda\varepsilon)}{\varepsilon}+C_2\frac{\cos(\lambda\varepsilon)}{\varepsilon} \tag{16}$$

Solving the Eq. (14b) yields what follows:

If $1-4\tau'\lambda^2>0$ then

$$B(\tau)=e^{\frac{-\tau}{2\tau'}}\left(C_3\sinh\frac{\kappa\tau}{2\tau'}+C_4\cosh\frac{\kappa\tau}{2\tau'}\right) \tag{17a}$$

If $1-4\tau'\lambda^2<0$ then

$$B(\tau)=e^{\frac{-\tau}{2\tau'}}\left(C_3\sin\frac{\kappa_i\tau}{2\tau'}+C_4\cos\frac{\kappa_i\tau}{2\tau'}\right) \tag{17a}$$

Where

$$\kappa=\sqrt{1-4\tau'\lambda^2} \tag{18a}$$

$$\kappa_i=i\kappa \tag{18b}$$

Using Eqs. (15a-c) yields to the following:

$$A(\varepsilon)=C_1\frac{\sin(\lambda\varepsilon)}{\varepsilon} \tag{19a}$$

$$B(\tau)=Ce^{\frac{-\tau}{2\tau'}}\begin{cases}\frac{1}{\kappa}\sinh\frac{\kappa\tau}{2\tau'}+\cosh\frac{\kappa\tau}{2\tau'}, & \kappa \text{ is real}\\ \frac{1}{\kappa_i}\sin\frac{\kappa_i\tau}{2\tau'}+\cos\frac{\kappa_i\tau}{2\tau'}, & \kappa \text{ is not real}\end{cases} \tag{19b}$$

Therefore, we obtain the solution of the unsteady problem as follows:

$$\psi(\varepsilon,\tau)=\sum_{f=1}^{F}C_f e^{\frac{-\tau}{2\tau'}}\left(\frac{1}{\kappa}\sinh\frac{\kappa\tau}{2\tau'}+\cosh\frac{\kappa\tau}{2\tau'}\right)\frac{\sin(\lambda_f\varepsilon)}{\varepsilon}+\sum_{f=F+1}^{\infty}C_f e^{\frac{-\tau}{2\tau'}}\left(\frac{1}{\kappa_i}\sin\frac{\kappa_i\tau}{2\tau'}+\cos\frac{\kappa_i\tau}{2\tau'}\right)\frac{\sin(\lambda_f\varepsilon)}{\varepsilon} \tag{20}$$

Where $\kappa=\sqrt{1-4\tau'\lambda_f^2}$ and λ_f are the roots of $\cos(x)=0$.

Using the Eq. (10c) and the orthogonality condition, we find the following:

$$C_f=\frac{2(-1)^f}{f\,\pi} \tag{21}$$

Therefore, the final solution of the problem is what follows:

$$\theta(\varepsilon,\tau) = 1 + \sum_{f=1}^{F} \frac{2(-1)^f}{f\pi} e^{\frac{-\tau}{2\tau'}} \left(\frac{1}{\kappa}\sinh\frac{\kappa\tau}{2\tau'} + \cosh\frac{\kappa\tau}{2\tau'} \right) \frac{\sin(\lambda_f \varepsilon)}{\varepsilon}$$
$$+ \sum_{f=F+1}^{\infty} \frac{2(-1)^f}{f\pi} e^{\frac{-\tau}{2\tau'}} \left(\frac{1}{\kappa_i}\sin\frac{\kappa_i\tau}{2\tau'} + \cos\frac{\kappa_i\tau}{2\tau'} \right) \frac{\sin(\lambda_f \varepsilon)}{\varepsilon} \quad (22)$$

In the special case, when $\tau' = 0$, i.e. parabolic model of the heat conduction equation, the solution is expressed as follows [21]:

$$\theta(\varepsilon,\tau) = 1 + \sum_{m=1}^{\infty} \frac{2(-1)^m}{m\pi} e^{-\lambda_m^2 \tau} \frac{\sin(\lambda_m \varepsilon)}{\varepsilon} \quad (23)$$

Where λ_m is the root of $\cos(x) = 0$.

It is possible to show that the Eq. (22) can be reduced to the above mentioned expression when τ' is equal to zero, which is in compliance with the Fourier model of heat conduction.

2.2. Laplace transforms solution

Laplace transformation is a strong tool for solving ordinary differential equations. This method is elaborated within many mathematical textbooks and is essential for engineering problems [22]. By taking the Laplace transformation of Eq. (4) to remove the τ -dependent terms it can be expressed as follows:

$$\varphi_{\varepsilon\varepsilon} + \frac{2}{\varepsilon}\varphi_{\varepsilon} = \tau'\left(s^2\varphi - s\theta(\varepsilon,0) - \theta_\tau(\varepsilon,0)\right) + \left(s\varphi - \theta(\varepsilon,0)\right) \quad (24)$$

Using initial conditions (5c-d), the above equation is reduced to the following:

$$\varphi_{\varepsilon\varepsilon} + \frac{2}{\varepsilon}\varphi_{\varepsilon} - \left(\tau' s^2 + s\right)\varphi = 0 \quad (25)$$

Also, the boundary conditions (5a-b) are transformed to the following:

$$\varphi_{\varepsilon}(0,s) = 0 \quad (26a)$$

$$\varphi(1,s) = \frac{1}{s} \quad (26b)$$

For solving Eq. (25) and B.Cs. (26) we apply two methods. Firstly, the variational method is used and applying various orders an improvement on the accuracy of the non-dimensional temperature profiles is achieved. Secondly, by an asymptotic expansion for the solution of this problem in the large values of s the manner of non-dimensional temperature distribution can be specified in the small values of time.

2.2.1. Variational formula

As Arpaci [20] used the variational formulation in parabolic heat conduction equation, this method can be used in the hyperbolic heat conduction equation. Using this method for Eq. (25) yields the following:

$$\int_0^1 \left(\varphi_{\varepsilon\varepsilon} + \frac{2}{\varepsilon}\varphi_{\varepsilon} - \left(\tau' s^2 + s\right)\varphi \right) \delta\varphi \, \mathrm{d}\varepsilon = 0 \quad (27)$$

This evaluation is carried out bellow for the first fifth approximations in order to investigate the effects of degree of approximations on the non-dimensional temperature.

First-order approximation

The simplest polynomial profile that satisfies the transformed B.Cs. (26a-b) is the following:

$$\varphi = \frac{1}{s}\left(1 + a_0\left(1 - \varepsilon^2\right)\right) \quad (28)$$

Using this profile into Eq. (27), the following is obtained:

$$\int_0^1 \left(-2a_0 + \frac{2}{\varepsilon}\left(-2a_0\varepsilon\right) - \left(\tau' s^2 + s\right)\left(1 + a_0\left(1-\varepsilon^2\right)\right) \right) \left(1-\varepsilon^2\right)\delta a_0 \mathrm{d}\varepsilon = 0 \quad (29)$$

So

$$\left(\frac{2}{15}\left(\tau' s^2 + s\right) a_0 - 4a_0 - \frac{2}{3}\left(\tau' s^2 + s\right)\left(1 + a_0\right) \right)\delta a_0 = 0 \quad (30)$$

Since δa_0 is arbitrary, the quantity inside the brackets in Eq. (30) must be zero. Therefore, it yields the following:

$$a_0 = -\frac{5}{2}\frac{s\left(\tau' s + 1\right)}{2\tau' s^2 + 2s + 15} \quad (31)$$

And consequently

$$\varphi(\varepsilon,s) = \frac{1}{s}\left(1 - \frac{5}{2}\frac{s\left(\tau' s + 1\right)}{2\tau' s^2 + 2s + 15}\left(1 - \varepsilon^2\right)\right) \quad (32)$$

Tacking Laplace inverse transformation, we have the following equations:

$$\varphi(\varepsilon,\tau) = \frac{5}{4}\frac{e^{-\frac{\tau}{2\tau'}}\left(\varepsilon^2 - 1\right)}{\sqrt{1-30\tau'}} \left(\sqrt{1-30\tau'}\cosh\left(\frac{\tau\sqrt{1-30\tau'}}{2\tau'}\right) + \sinh\left(\frac{\tau\sqrt{1-30\tau'}}{2\tau'}\right) \right) + 1 \quad (33)$$

Second-order approximation

A higher degree polynomial profile that satisfies the transformed B.Cs (26a-b) is the following:

$$\varphi=\frac{1}{s}\left(1+\left(a_0+a_1\varepsilon^2\right)\left(1-\varepsilon^2\right)\right) \quad (34)$$

Using this profile into Eq. (27), the following is obtained:

$$\int_0^1\left(\begin{array}{l}2a_1-12a_1\varepsilon^2-2a_0+\frac{2}{\varepsilon}\left(\begin{array}{l}2a_1\varepsilon\left(1-\varepsilon^2\right)-\\ \left(2\varepsilon\left(a_0+a_1\varepsilon^2\right)\right)\end{array}\right)-\\ \left(\tau' s^2+s\right)\left(1+\left(a_0+a_1\varepsilon^2\right)\left(1-\varepsilon^2\right)\right)\end{array}\right) \quad (35)$$

$$\left(1-\varepsilon^2\right)\left(\delta a_0+\varepsilon^2\delta a_1\right)\ \mathrm{d}\varepsilon=0$$

So

$$\left(-\frac{2}{3}\tau' s^2-\frac{8}{15}sa_0-\frac{2}{3}s+\frac{4}{3}a_1-\frac{8}{105}sa_1-\frac{8}{15}\tau' s^2 a_0-4a_0-\frac{8}{105}\tau' s^2 a_1\right)\delta a_0+$$
$$\left(-\frac{8}{315}\tau' s^2 a_1-\frac{8}{105}\tau' s^2 a_0-\frac{8}{315}sa_1-\frac{8}{105}sa_0-\frac{2}{15}s-\frac{2}{15}\tau' s^2-\frac{12}{35}a_1-\frac{4}{5}a_0\right)\delta a_1=0 \quad (36)$$

Since δa_0 and δa_1 are arbitrary, the quantities inside the two brackets in Eq. (36) must be zero. Therefore, the following equations are obtained:

$$a_0=-\frac{7}{8}\frac{s\left(\tau'^2 s^3+2\tau' s^2+\left(60\tau'+1\right)s+60\right)}{\tau'^2 s^4+2\tau' s^3+\left(42\tau'+1\right)s^2+42s+315} \quad (37a)$$

$$a_1=-\frac{21}{8}\frac{s^2\left(\tau'^2 s^2+2\tau' s+1\right)}{\tau'^2 s^4+2\tau' s^3+\left(42\tau'+1\right)s^2+42s+315} \quad (37b)$$

These equations are introduced in Eq. (34) using Maple software to take the Laplace inverse the second-order approximate solution is found and plotted.

Higher-order approximations

In order to increase the accuracy of approximation, we will use the following general form:

$$\varphi=\frac{1}{s}\left(1+\left(1-\varepsilon^2\right)\sum_{i=0}^{n-1}a_i\varepsilon^{2i}\right) \quad (38)$$

Which can be used as the nth-order of the approximation. Similar calculations for the cases $n=3$, 4 and 5 are done and the results are used in the subsequent analysis.

2.2.2. Using asymptotic expansion

The analytical solution of the transformed problem, Eq. (25) is readily solved to yield the following explicit formula:

$$\varphi(\varepsilon,s)=A(s)\frac{\sinh\left(\sqrt{\tau' s^2+s}\ \varepsilon\right)}{\varepsilon}+B(s)\frac{\cosh\left(\sqrt{\tau' s^2+s}\ \varepsilon\right)}{\varepsilon} \quad (39)$$

Using the transformed B.Cs. (26a-b) respectively, the following is obtained:

$$B(s)=0 \quad (40a)$$

$$A(s)=\frac{1}{s\ \sinh\left(\sqrt{\tau' s^2+s}\right)} \quad (40b)$$

Subsequently, the solution of the transformed equation together with boundary conditions is the following:

$$\varphi(\varepsilon,s)=\frac{1}{s\ \varepsilon}\frac{\sinh\left(\sqrt{\tau' s^2+s}\ \varepsilon\right)}{\sinh\left(\sqrt{\tau' s^2+s}\right)} \quad (41)$$

The Eq. (41) is non-linear in s variable and is too complex to take the inverse transformation. If we consider the special case of small values of dimensionless time τ corresponding to large values of the parameter s and use the geometrical series, the Eq. (41) for large values of s will be reduced into the following equation:

$$\varphi(\varepsilon,s)=\frac{1}{s\ \varepsilon}\frac{e^{\sqrt{\tau' s^2+s}\ \varepsilon}+e^{-\sqrt{\tau' s^2+s}\ \varepsilon}}{e^{\sqrt{\tau' s^2+s}}\left(1+e^{-2\sqrt{\tau' s^2+s}}\right)}=\frac{1}{s\ \varepsilon}\left(e^{\sqrt{\tau' s^2+s}\ \varepsilon}+e^{-\sqrt{\tau' s^2+s}\ \varepsilon}\right)\sum_{n=0}^{\infty}e^{-2n\sqrt{\tau' s^2+s}} \quad (42)$$

Which converges rapidly [21]. This special case is valid only for small values of dimensionless time τ, corresponding to large values of the parameter s. To find the inverse Laplace transform of this equation, it is found that using asymptotic series to expand the argument of the Eq. (42) to few initial terms, then summing the initial terms of the same expression, it is possible to find the inverse Laplace transform. It should be noticed that the accuracy is sufficient only for small values of times. The calculations are performed using three terms of the series by Maple software and the results are shown in the subsequent analysis.

3. Results and Discussion

To evaluate the degree of accuracy of variational method we compare the non-dimensional temperature with the non-dimensional radius for different values of non-dimensional relaxation times at different values of non-dimensional time, in Figs. 1-3. To present the results, due to the complexity of the formulations with different methods, two different softwares namely Maple and Matlab are used. An overview of these figures shows that the results of the variational method have a good agreement with the exact one, except for the case that ε is small while τ' and τ are large, simultaneously. In fact, higher values of τ' means the wavy nature of the heat propagation is stronger and therefore where the ε is small, i.e. in the points that are near $\varepsilon = 0$ which the boundary condition is implied on the derivation of the temperature instead of its value, considerable errors are occurred (see Fig. 3 for $\tau = 1$). It is seen that in each relaxation time as the order of approximation increases the result of the approximate solution becomes closer to the exact solution, especially, as the order of the approximation increases the location of the sharp discontinuity is evaluated with higher precision. Moreover, in each fixed τ' as the value of the time increases the result of all approximations has almost equal behavior, independent of the degree of the orders. As expected from the nature of the hyperbolic heat conduction, for all of the relaxation times and small values of time there are many points that have not touch the thermal wave. Also, in each fixed τ , the smaller value of relaxation time causes the approximate solution for each order has a more accurate result. It should be noted that a large value of relaxation time corresponds to a small value of propagation speed. In addition, as the value of relaxation time increases, the time required to reach the final temperature value increases. From these figures one can see that as the relaxation time increases the thermal waves become stronger and the accumulation of the energy behind the thermal wave increases and consequently the object peak temperature elevates. However, it is clear from Fig 3. that although for large values of ε the results of all approximations are close to each other, for small values of ε as the order of approximation increases from 1 to 3 the accuracy of the results increases and increasing the order of approximation from 3 to 5 causes that the accuracy of the solution reduces significantly.

Fig. 4 is plotted for investigating the effect of non-dimensional time on each order of approximate solution. By paying attention to the increase of the time, the nature of the thermal wave propagation, i.e. interaction and reflection can be seen, for this kind of heat conduction equation. For large values of non-dimensional time, $\tau = 10$ in this case, the non-dimensional temperature for all of the approximate solutions tends to approximate the final value. The thermal wave propagation is more obvious as the order of the approximation increases.

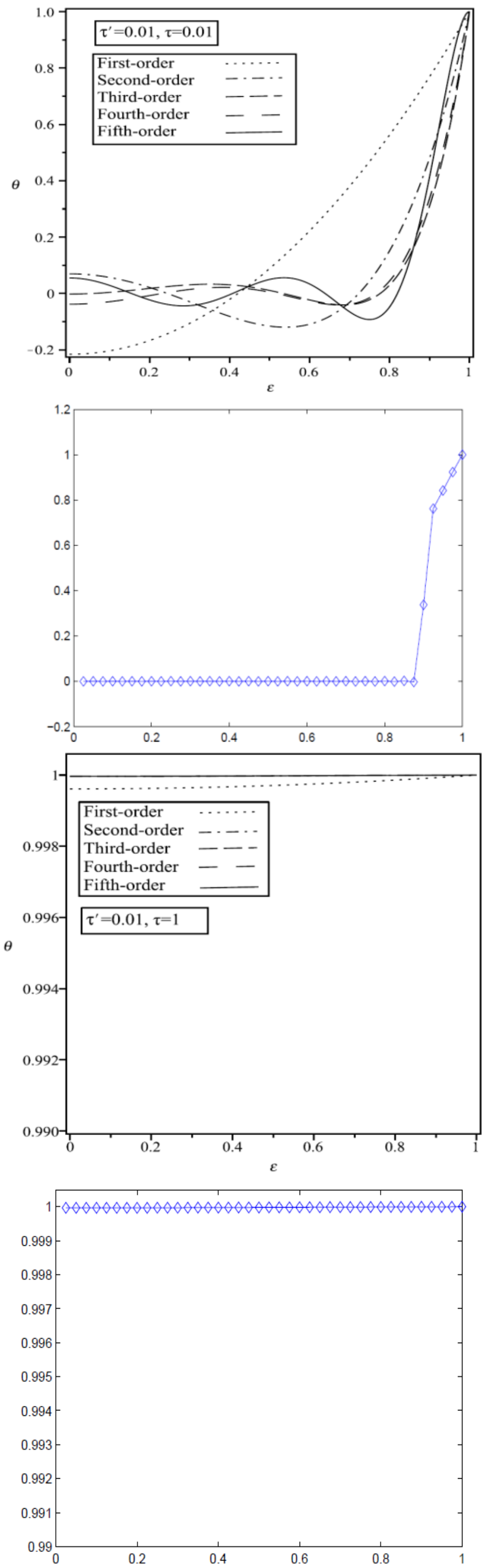


Fig. 1 The non-dimensional temperature for non-dimensional relaxation time $\tau' = 0.01$ and various values of non-dimensional time, $\tau = 0.01$ and 1.

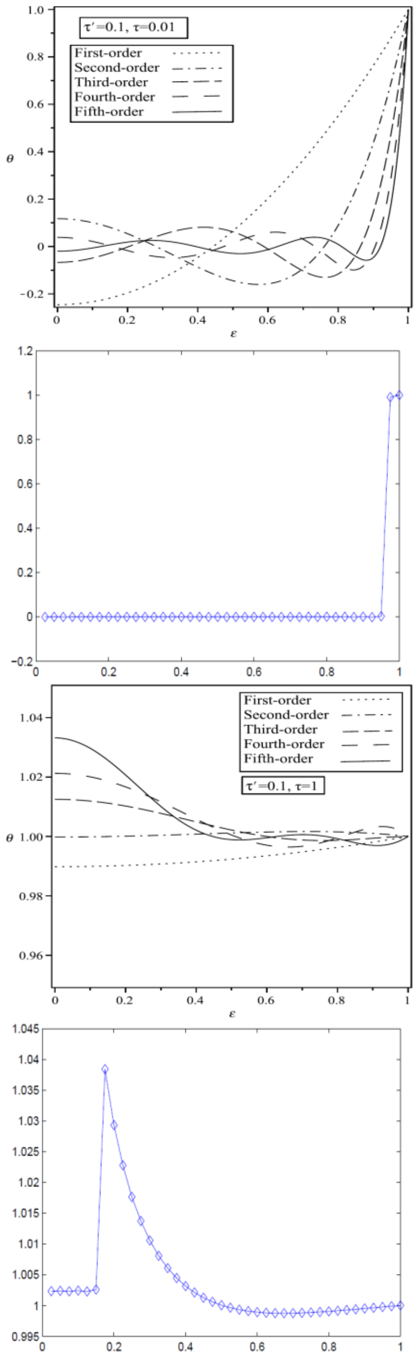
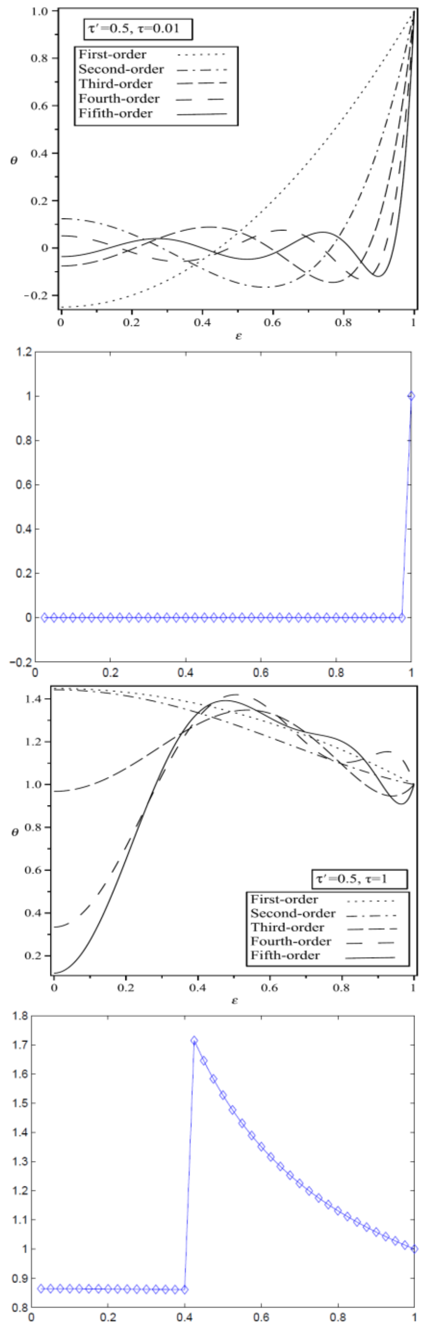


Fig. 2 The non-dimensional temperature for non-dimensional relaxation time $\tau' = 0.1$ and various values of non-dimensional time, $\tau = 0.01$ and 1.

A comparison of results obtained by several optional orders in variational method, asymptotic method and the exact solution for non-dimensional temperature is shown in Fig. 5. These calculations are presented for three non-dimensional radius $\varepsilon = 0.1$,0.5 and 0.9 and non-dimensional relaxation time $\tau' = 0.5$. Before any discussion, it must be noticed that the result of the asymptotic method is valid only for small values of time, i.e. as it can be seen from Fig. 5 for $\varepsilon = 0.1$, and the solution of the asymptotic expansion deviates from the correct result for the large time, in this case for $\tau = 0.8$. Although, as ε increases the solutions that are obtained from asymptotic expansion are valid for wider domain of τ . It is seen that for small and large values of ε which means the closer points to the boundary conditions, the precision of the variational method decreases. Also, it is clear as the order of approximation in the variational method increases the maximum value of the thermal wave and the time required to reach this phenomena to a special point are evaluated with more accuracy.



. 3 The non-dimensional temperature for non-dimensional relaxation time $\tau' = 0.5$ and various values of non-dimensional time, $\tau = 0.01$ and 1.

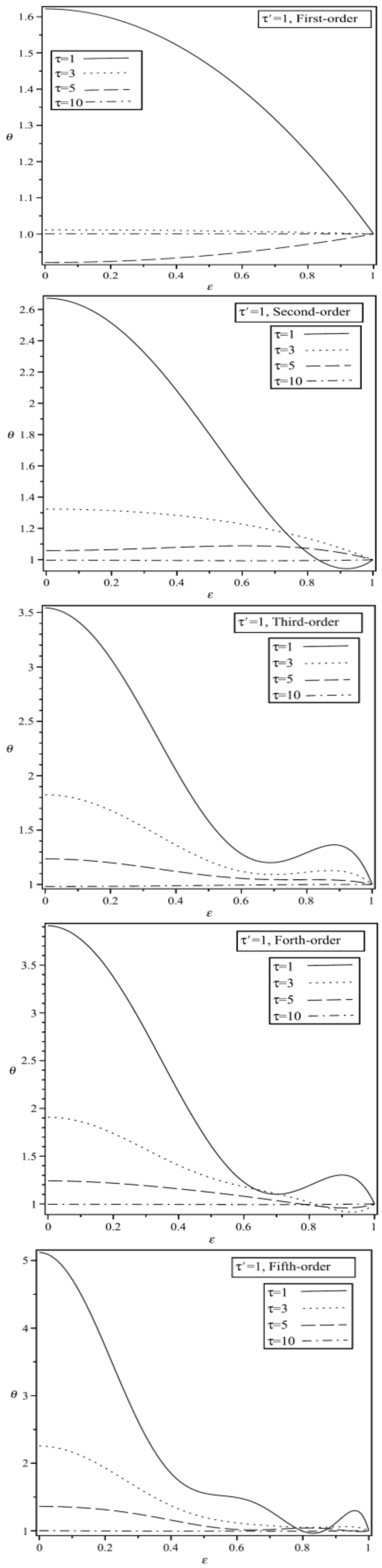


Fig. 4 The non-dimensional temperature for various values of non-dimensional time, $\tau = 1$,3 ,5 and 10.

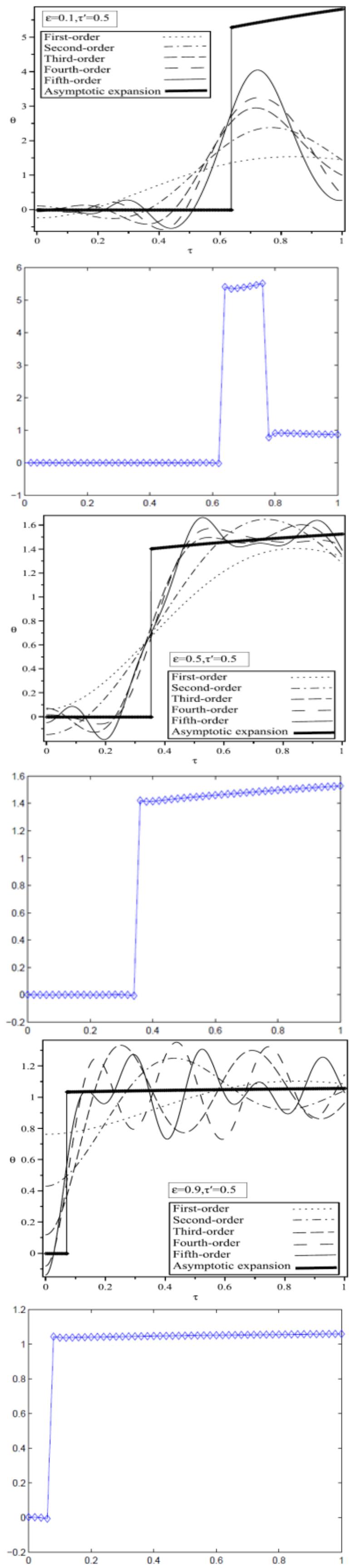


Fig. 5 The Comparison between the obtained results for variational method and Laplace inverse approximate solution (left-hand figures) and exact solution (right-hand figures)

4. Conclusion

Applying the separation of variables to a problem of hyperbolic heat conduction in a sphere an exact solution is derived. Then, using the Laplace transformation the problem is expressed in Laplace domain. For obtaining inverse Laplace two approximate approaches are used which are variational method and asymptotic expansion. A comparison of the results reveals the restriction and benefits of the variational formulation and asymptotic expansion for the Laplace inverse transformation. It has been shown that, as expected, the higher the order of the variational method the more similar are the results compared to the exact solution. Moreover, the asymptotic method was very much similar to the exact solution for the used parameters in this study.

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