A Study of a Stefan Problem Governed With Space–Time Fractional Derivatives

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1. Introduction

In recent years, many researchers have used fractional derivatives in various mathematical models due to their applicability in different fields of science and engineering [1-4]. It is well known that a fractional derivative is a good tool for taking into account the memory mechanism, particularly in some diffusive processes [5]. Stefan problems (moving boundary problems) with fractional derivatives [6-10] are typical problems from a mathematics point of view because of their nonlinear nature and the presence of a moving interface. Some exact solutions to Stefan problems can be seen in [8], [11], and [12]. Exact solutions to such problems are limited. Therefore, several approximate analytical methods [13-17] have been used to solve the Stefan problems with fractional derivatives. The approximate analytical method used in this literature is the optimal homotopy asymptotic method (OHAM).

The OHAM was developed by Marinca et al. [18], and it has been applied to solve a wide range of nonlinear differential equations [19-23]. Ghoreishi et. al. [24] presented the comparison between the homotopy analysis method and the OHAM for nonlinear age-structured population models. In 2013, Dinavand and Hosseini [25] also used the OHAM to investigate the temperature distribution equation in a convective straight fin with temperature-dependent thermal conductivity and the convective-
radiative cooling of a lumped system with variable specific heat.

This paper presents a mathematical model for a Stefan problem \[12\] with a space–time fractional derivative. In this model, the OHAM is used to find the expression of the temperature distribution in a given domain and location of a moving interface with the help of the Taylor series \[13\]. The obtained results are compared with the existing exact solution for a standard case and are in good agreement. An approachable analysis for a fractional case is also discussed.

2. Mathematical formulation

In this section, a mathematical model of a one-dimensional Stefan problem with a variable latent heat term \[12\] is considered. For this problem, we present a fractional model that involves space–time fractional derivatives, as given in \[11\]. The governing equations are as follows:

\[
D_\alpha^\beta T = \nu \frac{\partial^2}{\partial x^2} (D_\alpha^2 T), \quad 0 < x < s(t), \quad t > 0,
\]

\[
k D_\alpha^\beta T(x, 0, t > 0) = -b t^{(n-1)/2},
\]

\[
T(s(t), t) = 0, \quad t > 0,
\]

\[
k D_\alpha^\beta (T(s(t), t)) = -\gamma(s(t))^n D_\beta^\gamma s(t),
\]

where \( T(x, t) \) is the temperature distribution, \( s(t) \) is the moving interface, \( k \) is thermal conductivity, \( \nu \) is the thermal diffusion coefficient, \( b \) is a constant (\( b > 0 \) for melting, \( b < 0 \) for freezing), \( \gamma(s(t))^n \) is the variable latent heat per unit volume, and \( n \) is a non-negative integer. The operators \( D_\alpha^\beta \) and \( D_\alpha^\gamma \) are the Caputo fractional derivatives \[11,13\], which are defined as

\[
D_\alpha^\beta g(t) = D_t^{-\alpha+n} [g^{(n)}(t)], \quad (n-1 < \text{Re}(\alpha) \leq n, \quad n \in \mathbb{N}),
\]

\[
D_t^{-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} g(\tau) d\tau \quad (\alpha > 0),
\]

where \( \Gamma \) is the Gamma function.

In this paper, the following properties of fractional derivatives \[13-14\] are used:

(a) \( D_\alpha^\beta (z) = 0 \),

(b) \( D_\alpha^\beta D_\beta^\gamma = \frac{\Gamma(1+\beta) \Gamma(\beta-\alpha)}{\Gamma(1+\beta-\alpha)} D_\alpha^{\beta-\alpha} \),

where \( 0 \leq m \leq \alpha < m + 1, \quad \beta > m, \quad m \in \mathbb{N} \) and \( D_\alpha^\beta \) is the Caputo fractional derivative of \( \rho^\beta \).

3. Solution of the problem

First, Eqs. (1)–(4) are written in operator form as follows:

\[
vL(T(x, t)) - N(T(x, t)) = 0,
\]

\[
B \left( T, \frac{\partial T}{\partial x} \right) = 0,
\]

where \( L = \frac{\partial}{\partial t} \) is a linear operator, \( N = \frac{\partial^\beta}{\partial x^\beta} \) is a nonlinear operator, and \( B \) is a boundary operator.

According to the OHAM \[16, 21\], we construct an optimal \( T(x, t, p): [0, s(t)] \times [0, 1] \rightarrow R \), which satisfies

\[
(1 - p) vL(T(x, t, p)) + H(p) [vL(T(x, t, p)) - N(T(x, t, p))] = 0,
\]

where \( p \in [0, 1] \) is an embedding parameter, \( T(x, t, p) \) is an unknown function, and \( H(p) \) is a nonzero auxiliary function for \( p \neq 0 \) and \( H(0) = 0 \). Obviously, if \( p = 0 \),

\[
T(x, t; 0) = T_0(x, t),
\]

and when \( p = 1 \), then

\[
T(x, t; 1) = T(x, t).
\]

Clearly, as \( p \) increases from 0 to 1, the unknown function \( T(x, t, p) \) varies from \( T_0(x, t) \) to the solution \( T(x, t) \).

Now, we choose the auxiliary function \( H(p) \) in the following form:

\[
H(p) = p c_1 + p^2 c_2 + p^3 c_3 + \cdots,
\]

where \( c_1, c_2, c_3, \ldots \) are constants to be determined later.

The solution to Eq. (11) is considered in the following series form:

\[
T(x, t; p, c_i) = \sum_{k=0}^{\infty} \frac{T_k}{k!} (x, t, c_i) p^k, \quad i = 0, 1, 2, \ldots \ n,
\]

where \( 0 \leq m \leq \alpha < m + 1, \quad \beta > m, \quad m \in \mathbb{N} \) and \( D_\alpha^\beta \) is the Caputo fractional derivative of \( \rho^\beta \).
and
\[ s(t) = \sum_{n=0}^{\infty} p^n s_n(t), \quad (17) \]
where \( c_0 = 0 \) and \( T_0(x,t,0) = T_0(x,t) \).

Now, we expand the nonlinear term \( N(T(x,t;p,c_j)) \) into the following series form (as given in [24]):
\[ N(T(x,t;p,c_j)) = N_0(T_0) + \sum_{n=1}^{\infty} N_n(T_0,T_1,T_2,\cdots T_n)p^n, \quad (18) \]
where \( j = 1, 2, \cdots \).

Now, by substituting Eqs. (16) and (18) into Eq. (11) and equating the coefficients of like powers of \( p \), the following problems are obtained:
\[ p^0: \quad L(T_0(x,t)) = 0, \quad (19) \]
\[ p^1: \quad \nu L(T_1(x,t)) = -c_1 N_0(T_0(x,t)), \quad (20) \]
\[ p^2: \quad \nu L(T_2(x,t)) - \nu L(T_1(x,t)) = c_1 \nu L(T_1(x,t)) - c_2 N_0(T_0(x,t)) - c_1 N_1(T_0(x,t),T_1(x,t)), \quad (21) \]
and the general equation for \( T_k (x,t) \) is given as
\[ \nu L(T_k(x,t)) = \nu L(T_{k-1}(x,t)) - c_1 N_{k-1}(T_0(x,t)) + \sum_{i=1}^{k-1} c_i [\nu L(T_{i-1}(x,t)) - N_{i-1}(T_0(x,t),T_1(x,t),\cdots T_{i-1}(x,t))], \quad (22) \]
where \( k = 2, 3, \cdots \).

Substituting Eqs. (16) and (17) into the boundary conditions of (6) and (7) provides the following:
\[ k D_x^\alpha \left( \sum_{n=0}^{\infty} T_n(x=0,t,c_i) \right) p^n = -b_t^{(n-1)/2}, \quad (23) \]
and
\[ \sum_{n=0}^{\infty} T_n \left( \sum_{m=0}^{\infty} p^m s_n(t), c_i \right) p^n = 0, \quad (24) \]
where \( i = 0, 1, 2, \cdots n \).

The comparison of various powers of \( p \) can be shown by expanding \( T_l(x,t) \) in Taylor series form [13, 14] at a point, \( (s_0,t) \), as follows:
\[ T_l(x,t,c_i) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n T_l(s_0,t,c_i)}{\partial x^n} (x-s_0)^n, \quad (25) \]
where \( l = 0, 1, 2, 3, \cdots \) and \( i = 0, 1, 2, 3, \cdots, l \).

Eqs. (24) and (25) provide the following:
\[ \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{p^m}{m!} \left( \sum_{n=0}^{\infty} p^n s_n(t) \right)^m \frac{\partial^m}{\partial x^m} T_l(s_0,t,c_i) = 0. \quad (26) \]

The interface condition (4) becomes
\[ \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{p^m}{m!} \left( \sum_{n=0}^{\infty} p^n s_n(t) \right)^m \frac{\partial^m}{\partial x^m} T_l(s_0,t,c_i) = -\gamma^l \left( \sum_{n=0}^{\infty} p^n (s_n(t)) \right)^l D_\beta^l \left( \sum_{n=0}^{\infty} \frac{p^m}{m!} (s_m(t)) \right). \quad (27) \]

By considering Eq. (19) and comparing the coefficients of the power of \( p^0 \) from Eqs. (23), (26), and (27), the following system can be obtained:
\[ \frac{\partial}{\partial x} \left( D_x^\alpha T_0(x,t) \right) = 0, \quad (28) \]
\[ k D_x^\alpha \left( T_0(x=0,t) \right) = -b_t^{(n-1)/2}, \quad T_0(s_0,t) = 0, \]
\[ k \frac{\partial^\alpha}{\partial x^\alpha} T_0(s_0,t) = -\gamma (s_0(t))^n D_\beta^l (s_0(t)). \]

Taking Eq. (20) and comparing the coefficients of power for \( p^1 \) from Eqs. (23), (26), and (27) provides the following:
\[ \nu \frac{\partial}{\partial x} \left( D_x^\alpha T_1(x,t,c_i) \right) = c_1 D_\beta^l (T_0(x,t)) \]
\[ D_x^\alpha \left( T_0(0,t) \right) = 0, \quad T_1(s_0,t,c_i) + \gamma \frac{\partial T_0(s_0,t)}{\partial x^\alpha} = 0, \quad (29) \]
\[ \frac{\partial^\alpha}{\partial x^\alpha} T_1(s_0,t,c_i) + \gamma \frac{\partial^\alpha}{\partial x^\alpha} T_0(s_0,t) \]
\[ \frac{\partial^\alpha}{\partial x^\alpha} T_0(s_0,t) = -\gamma (s_0(t))^n D_\beta^l (s_0(t)). \]

Similarly, other systems can be found by comparing various powers of \( p \).

The solutions of the zeroth-order problem (28) are calculated as the following:
\[ T_0(x,t) = \frac{b}{k \Gamma(1+\alpha)} (s_0^\alpha - x^\alpha)^{t^{(n-1)/2}}, \quad (30) \]
and
\[ s_0 = a_0 t^\theta, \quad (31) \]
Where \( \theta = \beta + (n-1)/2 \) \( n+1 \), \( a_0 = \left( \frac{b \Gamma(1+\theta-\beta)}{\Gamma(1+\theta)} \right)^{1/n} \).
Substituting $T_0$ and $s_0$ into the first-order problem (29) and using the above process obtains the following expressions of $T_1(x,t,c_i)$:

$$T_1(x,t,c_i) = \frac{ab}{kT(1+\alpha)} s_1 s_0^{-1} \frac{n-1}{2} t^{\frac{n-1}{2}}$$

$$+ \frac{c_1 b}{\nu k} m_1 \left( 1+\alpha \right) t^{\frac{n-1}{2}}$$

$$+ \frac{m_2}{\Gamma(2+2\alpha)} \left( \frac{1+2\alpha}{1+2\alpha} - x^{1+2\alpha} t^{\frac{n+1-2\alpha}{2}} \right),$$

where $m_1 = \frac{\Gamma(1+\frac{n-1}{2}+\alpha\theta)}{\Gamma(1+\frac{n-1}{2}+\alpha\theta-\beta)}$,

$m_2 = \frac{\Gamma(1+\frac{n-1}{2})}{\Gamma(1+\frac{n-1}{2}-\beta)}$,

and $m_3 = \frac{m_1 a_0}{\Gamma(1+\alpha) \Gamma(2+\alpha)}$.

The expression of $s_1(t)$ can be calculated as:

$$s_1(t) = a_1 t^{\phi}. \tag{33}$$

where $a_1 = \frac{-c_1 b e^{(1+\alpha-N)} \left( m_2 - m_2 \right)}{\nu \gamma \Gamma(1+\alpha) \left( \Gamma(1+\phi) + n \left( \Gamma(1+\theta) \Gamma(1+\theta-\beta) \right) \right)}$.

and $\phi = (1+\alpha-n)\theta + (n-1)/2$.

The approximate solution of the temperature distribution can be determined as

$$T(x,t) = T_0(x,t) + T_1(x,t,c_1) + T_2(x,t,c_1,c_2) + \cdots, \tag{34}$$

and an approximate solution of $s(t)$ is given as

$$s(t) = s_0(t) + s_1(t) + s_2(t) + \cdots. \tag{35}$$

In order to get the constants involved in Eq. (34) for the expression of $T(x,t)$, the least square method is used [24]. For this purpose, residual is defined as:

$$R(x,t,c_1,c_2,\cdots c_i) = \nabla L(T(x,t,c_1,c_2,\cdots c_i)) \nabla N(T(x,t,c_1,c_2,\cdots c_i)). \tag{36}$$
Table 2. Comparison between exact and approximate solution of \( s(t) \) at \( n = 1 \).

<table>
<thead>
<tr>
<th>( b )</th>
<th>( t )</th>
<th>Exact value of ( S(t) )</th>
<th>Approximate value of ( S(t) ) by OHAM</th>
<th>Error (%)</th>
</tr>
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<tr>
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<td>0.0</td>
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Tables 1–2 represent comparisons between the exact and approximate values of the phase front \( s(t) \)’s positions at particular times \( t \) for \( \alpha = \beta = 1.0 \) (standard motion). The tables clearly show that the approximate results are sufficiently accurate and in agreement with the existing exact solution [12] for standard motion.

Figs. 1 and 2 represent the dependence of phase front \( s(t) \)’s movement trajectory on the thermal diffusion coefficient \( \nu \) for \( n = 0 \) at \( \alpha = 0.25, \beta = 0.75 \) and \( \alpha = 0.5, \beta = 1.0 \), respectively. Figs. 1–4 portray that the interface’s movement increases with an increase in the value of
the thermal diffusion coefficient for fractional cases (nonclassical or non-Fickian), which is similar to the case of standard motion [12].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Plot of \( s(t) \) vs. \( t \) at \( \alpha = 0.5, \beta = 1.0 \) and \( n = 0 \).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{Plot of \( s(t) \) vs. \( t \) at \( \alpha = 0.5, \beta = 1.0 \) and \( n = 0 \).}
\end{figure}

Figs. 5–6 show a variation in \( s(t) \)'s path for a different value of \( b \) for a non-classical or non-Fickian case. From these figures, it is clear that the phase front's movement increases with an increase in the value of the constant \( b \); that is, the melting (or freezing) process becomes fast as the value of the constant \( b \) increases.

5. Conclusion

In this work, we considered a mathematical model that contains space–time fractional derivatives and time-dependent surface-heat flux. An approximate solution of the model was obtained by the OHAM. It was observed that the interface movement increases with an increase in the value of the thermal diffusion coefficient \( \nu \) as well as the constant \( b \) for a nonclassical or non-Fickian case. Moreover, it can be seen that the proposed technique is sufficiently accurate and efficient for solving Stefan problems. It also was observed that it is convenient for controlling and adjusting the convergence of the series solution through the control parameters \( c_i \) in the OHAM.

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References