Journal of Heat and Mass Transfer Research 3 (2016) 145-151

Semnan University

Journal of Heat and Mass Transfer Research

Journal homepage: http://jhmtr.journals.semnan.ac.ir

# A Study of a Stefan Problem Governed With Space–Time Fractional Derivatives

Rajeev*, M.S. Kushwaha, A.K. Singh

*Department of Mathematical Sciences,*
*Indian Institute of Technology (Banaras Hindu University),*
*Varanasi-221005, India*

**PAPER INFO**

**History:**

Submitted 2 June 2015
Revised 21 July 2015
Accepted 27 July 2016

**Keywords**:

Optimal homotopy asymptotic method
Stefan problem
moving interface
fractional derivatives

**ABSTRACT**

This paper presents a fractional mathematical model of a one-dimensional phase-change problem (Stefan problem) with a variable latent-heat (a power function of position). This model includes space–time fractional derivatives in the Caputo sense and time-dependent surface-heat flux. An approximate solution of this model is obtained by using the optimal homotopy asymptotic method to find the solutions of temperature distribution in the domain $0 \le x \le s(t)$ and interface's tracking or location. The results thus obtained are compared with existing exact solutions for the case of the integer order derivative at some particular values of the governing parameters. The dependency of movement of the interface on certain parameters is also studied.



DOI: 10.22075/jhmtr.2016.384

## 1. Introduction

In recent years, many researchers have used fractional derivatives in various mathematical models due to their applicability in different fields of science and engineering [1-4]. It is well known that a fractional derivative is a good tool for taking into account the memory mechanism, particularly in some diffusive processes [5]. Stefan problems (moving boundary problems) with fractional derivatives [6-10] are typical problems from a mathematics point of view because of their nonlinear nature and the presence of a moving interface. Some exact solutions to Stefan problems can be seen in [8], [11], and [12]. Exact solutions to such problems are limited. Therefore, several approximate analytical methods [13-17] have been used to solve the Stefan problems with fractional derivatives. The approximate analytical method used in this literature is the optimal homotopy asymptotic method (OHAM).

The OHAM was developed by Marinca et al. [18], and it has been applied to solve a wide range of nonlinear differential equations [19-23]. Ghoreishi et. al. [24] presented the comparison between the homotopy analysis method and the OHAM for nonlinear age-structured population models. In 2013, Dinarvand and Hosseini [25] also used the OHAM to investigate the temperature distribution equation in a convective straight fin with temperature-dependent thermal conductivity and the convective–

Corresponding Author: *Rajeev*, Department of Mathematical Sciences,Indian Institute of Technology (Banaras Hindu University),Varanasi-221005, India
Email: rajeevbhu.mac@gmail.com

radiative cooling of a lumped system with variable specific heat.

This paper presents a mathematical model for a Stefan problem [12] with a space–time fractional derivative. In this model, the OHAM is used to find the expression of the temperature distribution in a given domain and location of a moving interface with the help of the Taylor series [13]. The obtained results are compared with the existing exact solution for a standard case and are in good agreement. An approachable analysis for a fractional case is also discussed.

## 2. Mathematical formulation

In this section, a mathematical model of a one-dimensional Stefan problem with a variable latent heat term [12] is considered. For this problem, we present a fractional model that involves space–time fractional derivatives, as given in [11]. The governing equations are as follow:

$$D_t^{\beta} T = \nu \frac{\partial}{\partial x}\left(D_x^{\alpha} T\right), \quad 0 < x < s(t),\ t > 0, \tag{1}$$

$$k\, D_x^{\alpha} T\left(x = 0, t > 0\right) = -b t^{(n-1)/2}, \tag{2}$$

$$T\left(s(t), t\right) = 0, \qquad t > 0, \tag{3}$$

$$k\, D_x^{\alpha}\left(T(s(t), t)\right) = -\gamma\, (s(t))^n\, D_t^{\beta} s(t), \tag{4}$$

where $T(x,t)$ is the temperature distribution, $s(t)$ is the moving interface, $k$ is thermal conductivity, $\nu$ is the thermal diffusion coefficient, $b$ is a constant ( $b > 0$ for melting, $b < 0$ for freezing), $\gamma\,(s(t))^n$ is the variable latent heat per unit volume, and $n$ is an non-negative integer. The operators $D_t^{\beta}$ and $D_x^{\alpha}$ are the Caputo fractional derivatives [11,13], which are defined as

$$D_t^{\alpha} g(t) = D_t^{\alpha - n}[g^{(n)}(t)], \quad (n - 1 < \mathrm{Re}(\alpha) \le n,\ n \in N), \tag{5}$$

$$D_t^{-\alpha} g(t) = \int_0^t \frac{(t-\tau)^{\alpha - 1}}{\Gamma(\alpha)} g(\tau) d\tau \qquad (\alpha > 0), \tag{6}$$

where $\Gamma$ is the Gamma function.

In this paper, the following properties of fractional derivatives [13-14] are used:

(a) $D_t^{\alpha}(z) = 0,$ (7)

where $z$ is a constant.

(b) $D_t^{\alpha} t^{\beta} = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta - \alpha}$ , (8)

where $0 \le m \le \alpha < m+1,\ \ \beta > m,\ \ m \in N$ and $D_t^{\alpha}$ is the Caputo fractional derivative of $t^{\beta}$ .

## 3. Solution of the problem

First, Eqs. (1)–(4) are written in operator form as follows:

$$\nu\, L\left(T(x,t)\right) - N\left(T(x,t)\right) = 0, \tag{9}$$

$$B\left(T, \frac{\partial T}{\partial x}\right) = 0, \tag{10}$$

where $L\left(= \dfrac{\partial^{1+\alpha}}{\partial x^{1+\alpha}}\right)$ is a linear operator, $N\left(= \dfrac{\partial^{\beta}}{\partial t^{\beta}}\right)$ is a nonlinear operator, and $B$ is a boundary operator.

According to the OHAM [16, 21], we construct an optimal $T(x,t,p) : [0, s(t)] \times [0,1] \to R$ , which satisfies

$$(1-p)\left[\nu\, L\left(T(x,t,p)\right)\right] = H(p)\left[\nu\, L(T(x,t,p)) - N\left(T(x,t,p)\right)\right], \tag{11}$$

$$B\left(T(x,t,p), \frac{\partial T(x,t,p)}{\partial x}\right) = 0, \tag{12}$$

where $p \in [0,1]$ is an embedding parameter, $T(x,t;p)$ is an unknown function, and $H(p)$ is a nonzero auxiliary function for $p \ne 0$ and $H(0) = 0$. Obviously, if $p = 0$ ,

$$T(x,t;\,0) = T_0(x,t), \tag{13}$$

and when $p = 1$, then

$$T(x,t;\,1) = T(x,t). \tag{14}$$

Clearly, as $p$ increases from 0 to 1, the unknown function $T(x,t,p)$ varies from $T_0(x,t)$ to the solution $T(x,t)$ .

Now, we choose the auxiliary function $H(p)$ in the following form:

$$H(p) = p c_1 + p^2 c_2 + p^3 c_3 + \cdots, \tag{15}$$

where $c_{1,}\, c_{2,}\, c_3$........ are constants to be determined later.

The solution to Eq. (11) is considered in the following series form:

$$T(x,t;p,c_i) = \sum_{k=0}^{\infty} T_k\left(x,t,c_i\right) p^k,\ i = 0,1,2,\cdots n, \tag{16}$$

and

$$s(t)=\sum_{n=0}^{\infty}p^n s_n(t)\,, \tag{17}$$

where $c_0=0$ and $T_0(x,t,0)=T_0(x,t)$.

Now, we expand the nonlinear term $N(T(x,t;p,c_j))$ into the following series form (as given in [24]):

$$N(T(x,t;p,c_j))=N_0(T_0)+\sum_{m\geq 1}N_m\left(T_0,T_1,T_2,\cdots T_m\right)p^m, \tag{18}$$

where $j=1,\ 2,\cdots$.

Now, by substituting Eqs. (16) and (18) into Eq. (11) and equating the coefficients of like powers of $p$, the following problems are obtained:

$$p^0:\quad L(T_0(x,t))=0\ , \tag{19}$$

$$p^1:\quad \nu\ L(T_1(x,t))=-c_1N_0\left(T_0(x,t)\right), \tag{20}$$

$$\begin{aligned}p^2:\ &\nu\,L(T_2(x,t))-\nu\,L(T_1(x,t))=c_1\nu\,L(T_1(x,t))\\ &-c_2\,N_0(T_0(x,t))-c_1\,N_1\left(T_0(x,t),T_1(x,t)\right),\end{aligned} \tag{21}$$

and the general equation for $T_k(x,t)$ is given as

$$\begin{aligned}&\nu\,L(T_k(x,t))=\nu\,L(T_{k-1}(x,t))-c_k\,N_0(T_0(x,t))\\ &+\sum_{i=1}^{k-1}c_i\left[\nu\,L(T_{k-i}(x,t))-N_{k-i}\left(T_0(x,t),T_1(x,t),\cdots T_{k-1}(x,t)\right)\right],\end{aligned} \tag{22}$$

where $k=2,3,\cdots$ .

Substituting Eqs. (16) and (17) into the boundary conditions of (6) and (7) provides the following:

$$k\,D_x^{\alpha}\left(\sum_{n=0}^{\infty}T_n\left(x=0,t,c_i\right)\right)p^n=-bt^{(n-1)/2}\ , \tag{23}$$

and

$$\sum_{n=0}^{\infty}T_n\left(\sum_{n=0}^{\infty}p^n s_n\,,t,c_i\right)p^n=0\,, \tag{24}$$

where $i=0,1,\ 2,\cdots n$.

The comparison of various powers of $p$ can be shown by expending $T_i(x,t)$ in Taylor series form [13, 14] at a point, $(s_0,t)$ , as follows:

$$T_l\,(x,t,c_i)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n T_l\,(s_0\,,t,c_i)}{\partial x^n}(x-s_0)^n\,, \tag{25}$$

where $l=0,1,2,3\cdots$ and $i=0,1,2,3\cdots,l$.

Eqs. (24) and (25) provide the following:

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{p^l}{m!}\left(\sum_{n=1}^{\infty}p^n s_n\right)^m\frac{\partial^m}{\partial x^m}T_l\,(s_0\,,t,c_i)=0\,. \tag{26}$$

The interface condition (4) becomes

$$\begin{aligned}&\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{p^l}{m!}\left(\sum_{n=1}^{\infty}p^n s_n(t)\right)^m\frac{\partial^{m+\alpha}}{\partial x^{m+\alpha}}T_l\,(s_0\,,t,c_i)\\ &=-\gamma\left(\sum_{m=0}^{\infty}p^m(s_m\,(t))\right)^n D_t^{\beta}\left(\sum_{m=0}^{\infty}p^m(s_m\,(t))\right).\end{aligned} \tag{27}$$

By considering Eq. (19) and comparing the coefficients of the power of $p^0$ from Eqs. (23), (26), and (27), the following system can be obtained:

$$\begin{aligned}&\frac{\partial}{\partial x}\left(D_x^{\alpha}\,T_0(x,t)\right)=0,\\ &k\,D_x^{\alpha}\left(T_0(x=0,t)\right)=-bt^{(n-1)/2},\\ &T_0(s_0,t)=0,\\ &k\,\frac{\partial^{\alpha}T_0(s_0\,,t)}{\partial x^{\alpha}}=-\,\gamma\left(s_0(t)\right)^n D_t^{\beta}\left(s_0(t)\right).\end{aligned} \tag{28}$$

Taking Eq. (20) and comparing the coefficients of power for $p^1$ from Eqs. (23), (26), and (27) provides the following:

$$\begin{aligned}&\nu\frac{\partial}{\partial x}\left(D_x^{\alpha}\,T_1(x,t,c_1)\right)=c_1\,D_t^{\beta}\left(T_0(x,t)\right),\\ &D_x^{\alpha}\left(T_0(0,t)\right)=0,\\ &T_1(s_0\,,t,c_1)+s_1\frac{\partial T_0(s_0\,,t)}{\partial x}=0,\\ &\frac{\partial^{\alpha}T_1(s_0\,,t,c_1)}{\partial x^{\alpha}}+s_1\frac{\partial^{1+\alpha}T_0(s_0\,,t)}{\partial x^{1+\alpha}}=\\ &-\frac{\gamma}{k}\left(\left(s_0\right)^n D_t^{\beta}\left(s_1(t)\right)+n\left(s_0\right)^{n-1}s_1\,D_t^{\beta}\left(s_0\right)\right).\end{aligned} \tag{29}$$

Similarly, other systems can be found by comparing various powers of $p$ .

The solutions of the zeroth-order problem (28) are calculated as the following:

$$T_0(x,t)=\frac{b}{k\,\Gamma(1+\alpha)}\left(s_0^{\alpha}-x^{\alpha}\right)t^{(n-1)/2}\,, \tag{30}$$

and

$$s_0=a_0\,t^{\theta}\,, \tag{31}$$

Where $\theta=\dfrac{\beta+(n-1)/2}{n+1}$, $\quad a_0=\left(\dfrac{b\,\Gamma(1+\theta-\beta)}{\gamma\,\Gamma(1+\theta)}\right)^{\frac{1}{1+n}}$ .

Substituting $T_0$ and $s_0$ into the first-order problem (29) and using the above process obtains the following expressions of $T_1(x,t,c_1)$ :

$$\begin{aligned} T_1(x,t,c_1) &= \frac{\alpha b}{k\,\Gamma(1+\alpha)} s_1 s_0^{1-\alpha}\, t^{\frac{n-1}{2}} \\ &+ \frac{c_1 b}{\nu k}\left( m_3\left(x^{1+\alpha} - s_0^{1+\alpha}\right) t^{\alpha\theta+\frac{n-1}{2}-\beta} \right. \\ &\left. + \frac{m_2}{\Gamma(2+2\alpha)}\left(s_0^{1+2\alpha} - x^{1+2\alpha}\right) t^{\frac{n-1}{2}-\beta} \right), \end{aligned} \tag{32}$$

where $m_1 = \dfrac{\Gamma(1+\frac{n-1}{2}+\alpha\theta)}{\Gamma(1+\frac{n-1}{2}+\alpha\theta-\beta)}$ ,

$m_2 = \dfrac{\Gamma(1+\frac{n-1}{2})}{\Gamma(1+\frac{n-1}{2}-\beta)}$ ,

and $m_3 = \dfrac{m_1\, a_0^{\alpha}}{\Gamma(1+\alpha)\Gamma(2+\alpha)}$ .

The expression of $s_1(t)$ can be calculated as:

$$s_1(t) = a_1 t^{\phi}, \tag{33}$$

where $a_1 = \dfrac{-c_1\, b\, a_0^{(1+\alpha-n)}\left(m_1 - \dfrac{m_2}{(1+\alpha)}\right)}{\nu\gamma\Gamma(1+\alpha)\left(\dfrac{\Gamma(1+\phi)}{\Gamma(1+\phi-\beta)} + n\dfrac{\Gamma(1+\theta)}{\Gamma(1+\theta-\beta)}\right)}$ ,

and $\phi = (1+\alpha-n)\theta + (n-1)/2$ .

The approximate solution of the temperature distribution can be determined as

$$T(x,t) = T_0(x,t) + T_1(x,t,c_1) + T_2(x,t,c_1,c_2) + \cdots, \tag{34}$$

and an approximate solution of $s(t)$ is given as

$$s(t) = s_0(t) + s_1(t) + s_2(t) + \cdots . \tag{35}$$

In order to get the constants involved in Eq. (34) for the expression of $T(x,t)$, the least square method is used [24]. For this purpose, residual is defined as:

$$\begin{aligned} R(x,t;c_1,c_2,\cdots c_l) &= \nu\, L\big(T(x,t,c_1,c_2,\cdots c_l)\big) \\ &- N\big(T(x,t,c_1,c_2,\cdots c_l)\big). \end{aligned} \tag{36}$$

If $R(x,t;c_i)=0$, then $T(x,t;c_i)$ will be the exact solution. Generally, the OHAM gives an approximate solution. Therefore, in such a case, $R(x,t;c_i)\neq 0$, but the function can be minimized as

$$J(c_i) = \int_0^t \int_0^{s(t)} R^2(x,t;c_i)\, dx dt, \tag{37}$$

where R is the residual. The constants $c_i\ (i=1,2......,l)$ can be obtained optimally from the following conditions:

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \cdots = \frac{\partial J}{\partial c_l} = 0. \tag{38}$$

## 4. Numerical results and discussion

In this section, numerical results for interface position $s(t)$ are obtained with the help of Wolfram Research (8.0.0) software by considering only $c_1$, and the results are presented through tables and figures.

Table 1. Comparison between exact and approximate solution of $s(t)$ at $n$ = 0.

| $b$ | $t$ | Exact value of $S(t)$ | Approximate value of $S(t)$ by OHAM | Error (%) |
|---|---|---|---|---|
| 0.1 | 0 | 0.0 | 0.0 | 0.0 |
| | 5 | 0.4428497 | 0.4449844 | 0.4820 |
| | 10 | 0.6262841 | 0.6293030 | 0.4820 |
| | 15 | 0.7668507 | 0.7707735 | 0.5112 |
| | 20 | 0.8856994 | 0.8899689 | 0.4820 |
| | 25 | 0.9902421 | 0.9950155 | 0.4820 |
| 0.25 | 0 | 0.0 | 0.0 | 0.0 |
| | 1 | 0.4728215 | 0.4847701 | 2.5271 |
| | 2 | 0.6686706 | 0.6854703 | 2.5124 |
| | 3 | 0.8189509 | 0.8395262 | 2.5124 |
| | 4 | 0.9456430 | 0.9694014 | 2.5124 |
| | 5 | 1.0030059 | 1.0282054 | 2.5124 |
| 0.5 | 0 | 0.0 | 0.0 | 0.0 |
| | 0.25 | 0.4193648 | 0.4439988 | 5.8741 |
| | 0.50 | 0.5930714 | 0.6279091 | 5.8741 |
| | 0.75 | 0.7263611 | 0.7690284 | 5.8741 |
| | 1.00 | 0.8387296 | 0.8879975 | 5.8741 |
| | 1.30 | 0.9562989 | 1.0124729 | 5.8741 |

Table 2. Comparison between exact and approximate solution of $s(t)$ at $n$ = 1.

| $b$ | $t$ | Exact value of $S(t)$ | Approximate value of $S(t)$ by OHAM | Error (%) |
|---|---|---|---|---|
| 0.1 | 0 | 0.0 | 0.0 | 0.0 |
| | 1 | 0.4275253 | 0.4332152 | 1.3309 |
| | 2 | 0.6046120 | 0.6126588 | 1.3309 |
| | 3 | 0.7404955 | 0.7503507 | 1.3309 |
| | 4 | 0.8550505 | 0.8664304 | 1.3309 |
| | 5 | 0.9559756 | 0.9686986 | 1.3308 |
| 0.25 | 0 | 0.0 | 0.0 | 0.0 |
| | 0.5 | 0.4527556 | 0.4719994 | 4.2504 |
| | 1.0 | 0.6402931 | 0.6675079 | 4.2504 |
| | 1.5 | 0.7841957 | 0.8175269 | 4.2504 |
| | 2.0 | 0.9055112 | 0.9439988 | 4.2504 |
| | 2.5 | 1.0123923 | 1.0554227 | 4.2504 |
| 0.5 | 0.0 | 0.0 | 0.0 | 0.0 |
| | 0.25 | 0.4222513 | 0.4536318 | 7.4317 |
| | 0.5 | 0.5971534 | 0.6415322 | 7.4317 |
| | 0.75 | 0.7313607 | 0.7857133 | 7.4317 |
| | 1.0 | 0.8445026 | 0.9072635 | 7.4317 |
| | 1.25 | 0.9441826 | 1.0143515 | 7.4317 |

Tables 1–2 represent comparisons between the exact and approximate values of the phase front $s(t)$'s positions at particular times $t$ for $\alpha = \beta = 1.0$ (standard motion). The tables clearly show that the approximate results are sufficiently accurate and in agreement with the existing exact solution [12] for standard motion.

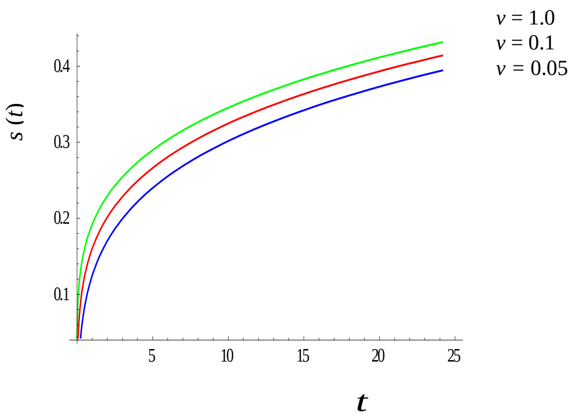


Fig. 1 Plot of $s(t)$ vs. $t$ at $\alpha = 0.25$, $\beta = 0.75$ and $n$ = 0

Figs. 1 and 2 represent the dependence of phase front $s(t)$'s movement trajectory on the thermal diffusion coefficient $\nu$ for $n = 0$ at $\alpha = 0.25$, $\beta = 0.75$ and $\alpha = 0.5$, $\beta = 1.0$, respectively.

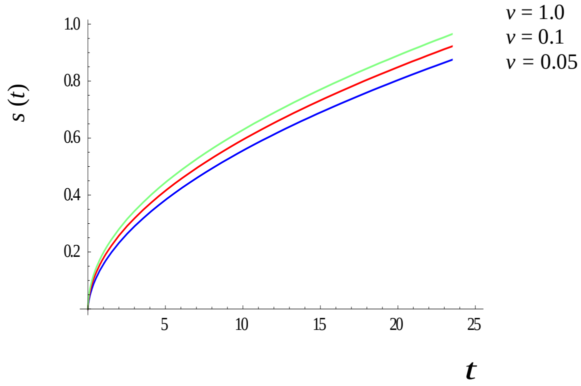


Fig. 2 Plot of $s(t)$ vs. $t$ at $\alpha = 0.5$, $\beta = 1.0$ and $n$ = 0.

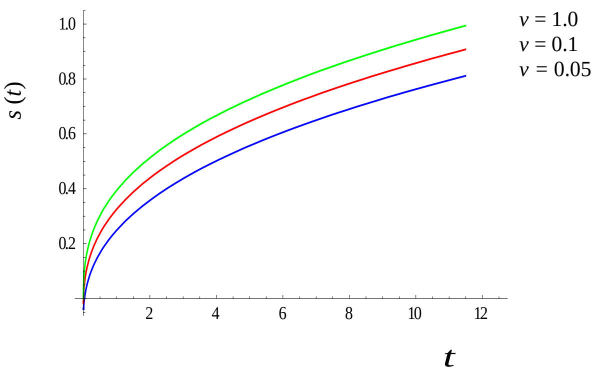


Fig. 3 Plot of $s(t)$ vs. $t$ at $\alpha = 0.25$ $\beta = 0.75$ $n$ = 1.

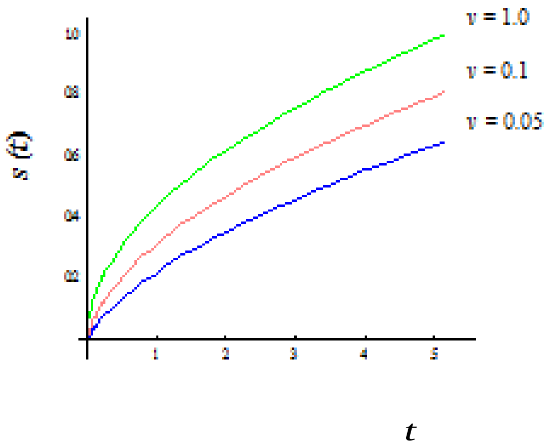


Fig. 4 Plot of $s(t)$ vs. $t$ at $\alpha = 0.5$, $\beta = 1.0$ and $n$ = 1.

Figs. 3 and 4 also depict the dependence of phase front $s(t)$'s path on the thermal diffusion coefficient $\nu$ for $n$ = 1 at $\alpha = 0.25$, $\beta = 0.75$ and $\alpha = 0.5$, $\beta = 1.0$, respectively. Figs. 1–4 portray that the interface's movement increases with an increase in the value of the thermal diffusion coefficient for fractional cases (nonclassical or non-Fickian), which is similar to the case of standard motion [12].

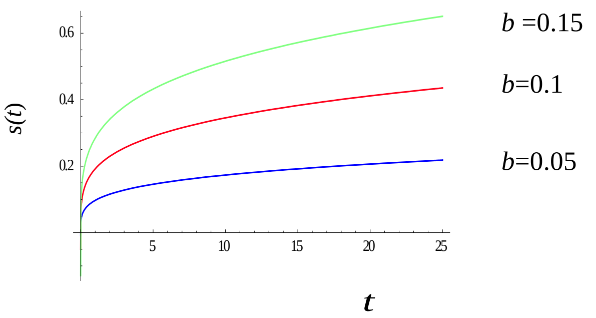


Fig. 5 Plot of $s(t)$ vs. $t$ at $\alpha = 0.5$, $\beta = 1.0$ and $n = 0$.

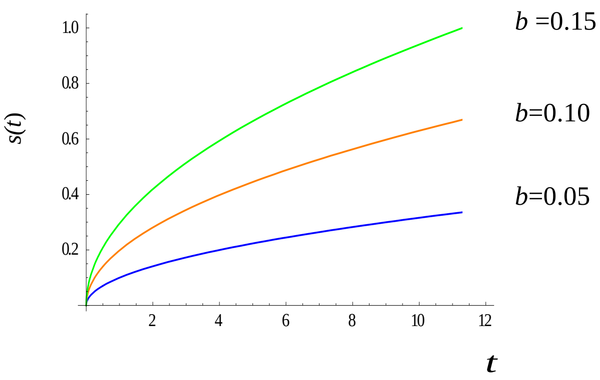


Fig. 6 Plot of $s(t)$ vs. $t$ at $\alpha = 0.5$, $\beta = 1.0$ and $n = 0$.

Figs. 5–6 show a variation in $s(t)$'s path for a different value of $b$ for a non-classical or non-Fickian case. From these figures, it is clear that the phase front's movement increases with an increase in the value of the constant $b$; that is, the melting (or freezing) process becomes fast as the value of the constant $b$ increases.

## 5. Conclusion

In this work, we considered a mathematical model that contains space–time fractional derivatives and time-dependent surface-heat flux. An approximate solution of the model was obtained by the OHAM. It was observed that the interface movement increases with an increase in the value of the thermal diffusion coefficient $v$ as well as the constant $b$ for a nonclassical or non-Fickian case. Moreover, it can be seen that the proposed technique is sufficiently accurate and efficient for solving Stefan problems. It also was observed that it is convenient for controlling and adjusting the convergence of the series solution through the control parameters $c_i$ in the OHAM.


## Acknowledgements

The authors express their sincere thanks to the anonymous referees for their valuable suggestions for the improvement of the paper.


## References


[1] A.S. Chaves, A fractional diffusion equation to describe Lévy flights. *Phys. Lett.* A, 239, 13–16 (1998).

[2] D.A. Benson, S.W. Wheatcraft, M. M. Meerschaert, The fractional-order governing equation of Lévy motion, Water Resources Res., 36, 1413–1423 (2000).

[3] Y. Aoki, M. Sen, S. Paolucci, Approximation of transient temperatures in complex geometries using fractional derivatives, Heat Transfer, 44, 771–777 (2008).

[4] H. Jiang, F. Liu, I. Turner, K. Burrage, Analytical solutions for the multi-term time-space Caputo-Riesz fractional advection-diffusion equations on a finite domain, Journal of Mathematical Analysis and Applications, 389, 1117-1127 (2012).

[5] Zˇ. Tomovskia, T. Sandev, R. Metzler, J. Dubbeldam, Generalized space–time fractional diffusion equation with composite fractional time derivative, Physica A, 391, 2527–42 (2012).

[6] X.C. Li, M. Y. Xu, S.W. Wang, Analytical solutions to the moving boundary problems with time–space fractional derivatives in drug release devices, J Phys A: Math. Theor., 40, 12131–12141 (2007).

[7] X.C. Li, M.Y. Xu, S.W. Wang, Scale-invariant solutions to partial differential equations of fractional order with a moving boundary condition, J Phys A: Math. Theor., 41, 155202 (2008).

[8] J. Liu, M. Xu, Some exact solutions to Stefan problems with fractional differential equations, J. Math Anal. Appl., 351, 536-542 (2009).

[9] C. J. Vogl, M. J. Miksis, S. H. Davis, Moving boundary problems governed by anomalous diffusion.Proc. R. Soc. A, 468, 3348-3369 (2012).

[10] S. Das, R. Kumar, P.K. Gupta, Analytical approximate solution of space–time fractional

diffusion equation with a moving boundary condition, Z. Naturforsch. A 66 a, 281–288 (2011).

[11] V.R. Voller, An exact solution of a limit case Stefan problem governed by a fractional diffusion equation, Int. J. Heat Mass Transfer., 53, 5622-5625 (2010).

[12] Y. Zhou, Y. Wang, W. Bu, Exact solution for a Stefan problem with latent heat a power function of position, International Journal of Heat and Mass Transfer, 69, 451–454 (2014).

[13] L. Xicheng, M. Xu, X. Jiang, Homotopy perturbation method to time-fractional diffusion equation with a moving boundary condition, Applied Mathematics and Computation, 208, 434–439 (2009).

[14] Rajeev, M.S. Kushwaha, Homotopy perturbation method for a limit case Stefan problem governed by fractional diffusion equation, Appl. Math Model, 37, 3589-3599 (2013).

[15] S. Das, Rajeev, Solution of fractional diffusion equation with a moving boundary condition by variational iteration method and Adomian decomposition method, Z Naturforsch . 65a, 793-799 (2010).

[16] R. Grzymkowski, D. Słota, One-phase inverse Stefan problem solved by Adomian decomposition method, Comput Math Appl., 51, 33-40 (2006).

[17] Rajeev, M. S. Kushwaha, A. Kumar, An approximate solution to a moving boundary problem with space–time fractional derivative in fluvio-deltaic sedimentation process, Ain Shams Engineering Journal, 4, 889–895 (2013).

[18] V. Marinca, N. Herisanu, Application of homotopy Asymptotic method for solving non-linear equations arising in heat transfer, Int. Comm. Heat

[19] N. Herisanu, V. Marinca, Accurate analytical solutions to oscillators with discontinuities and fractional-power restoring force by means of the optimal homotopy asymptotic method. Comput. Math. Appl. 60, 1607–1615 (2010).

[20] V. Marinca, N. Herisanu, Determination of periodic solutions for the motion of a particle on a rotating parabola by means of the optimal homotopy asymptotic method, J. Sound Vib. 329, 1450–1459 (2010).

[21] S. Iqbal, M. Idrees, A.M. Siddiqui, A.R. Ansari, Some solutions of the linear and nonlinear Klein–Gordon equations using the optimal homotopy asymptotic method, Appl. Math. Comput., 216, 2898–2909 (2010).

[22] S. Iqbal, A. Javed, Application of optimal homotopy asymptotic method for the analytic solution of singular Lane–Emden type equation, Appl. Math. Comput., 217, 7753–7761 (2011).

[23] M. S. Hashmi, N. Khan, S. Iqbal, Optimal homotopy asymptotic method for solving nonlinear Fredholm integral equations of second kind, Applied Mathematics and Computation, 218, 10982–10989 (2012).

[24] M. Ghoreishi , A.I.B. Md. Ismail , A.K. Alomari , A.S. Bataineh, The comparison between Homotopy Analysis Method and Optimal Homotopy Asymptotic Method for nonlinear age-structured population Models, Commun Nonlinear Sci. Numer. Simulat., 17, 1163–1177 (2012).

[25] S. Dinarvand, R. Hosseini, Optimal homotopy asymptotic method for convective–radiative cooling of a lumped system, and convective straight fin with temperature-dependent thermal conductivity, Afrika Matematika., 24, 103-116 (2013).